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Sobolev-Type Orthogonal Polynomials: The Nondiagonal Case

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Sobolev type orthogonal polynomials have been the object of increasing interest in the last few years. In this paper we introduce a generalization of the usual Sobolev-type inner product and we compare it with the strict diagonal case. Zeros and asymptotic properties of these kinds of polynomial sequence are studied.

1. INTRODUCTION

Let μ be a positive Borel measure on an interval (finite or infinite) $I \subset \mathbf{R}$ whose moments are finite and whose support is an infinite set. In the sequel (P_n) denotes the sequence of monic orthogonal polynomials (SMOP) with respect to μ and $\mathring{I}$ denotes the interior of I.

We denote by (Q_n) the SMOP associated with the inner product

$$(P, Q) = \int_I P(x)\, Q(x)\, d\mu + \mathbb{P}(c)^t A \mathbb{Q}(c) \tag{1.1}$$

where $A \in \mathbf{R}^{(r+1, r+1)}$ is a positive semidefinite matrix, $c \in \mathbf{R}$ and given a polynomial $P(x)$, $\mathbb{P}(x)$ denotes the matrix

$$\begin{pmatrix} P(x) \\ P'(x) \\ \vdots \\ P^{(r)}(x) \end{pmatrix}$$

and $\mathbb{P}(x)^t$ its transpose.

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The inner product (1.1) has been considered by several authors when A is diagonal (see, for instance, [2], [6], [9], [12], [14] and [16]). We are interested in the analysis of the inner product (1.1), not only as a generalization of the diagonal case but

(i) It appears as the natural form of an inner product such that the multiplication operator $H=(x-c)^{r+1}$ in the space of the polynomials is symmetric and commutes with the multiplication operator $G=(x-c)$ (see [5, Theorem 6]).

(ii) In [3] a particular case of the following inner product

$$\langle f, g\rangle = \int_{-a}^{a} fg\, d\mu + M[f(c)\,g(c) + f(-c)\,g(-c)] + N[f'(c)\,g'(c) + f'(-c)\,g'(-c)] \tag{1.2}$$

has been studied, where μ is a symmetric positive Borel measure on $I=[-a, a]$ and M, $N \in \mathbf{R}^+$. If (Q_n) is the SMOP with respect to (1.2), then $Q_{2n}(x)=U_n(x^2)$ and $Q_{2n+1}(x)=xV_n(x^2)$ hold. It follows that (U_n) and (V_n) are SMOP with respect to inner products like (1.1) with A diagonal and non-diagonal respectively (see [1]).

In this paper a comparison with the diagonal case is provided. In Section 2, several representation formulas of (Q_n) in terms of (P_n) are obtained, mainly when A is a 2×2 matrix. These results are basic for the development of the sections 3 and 4. The behaviour of the zeros in different situations is analyzed in Section 3.

In Section 4, a study of relative asymptotic properties for the orthonormal polynomials q_n in terms of the orthonormal polynomials p_n is presented. A comparison with the results by P. Nevai ([17]), F. Marcellán and W. Van Assche ([15]) and G. López, F. Marcellán and W. Van Assche ([11]) is given; our techniques are different and some new results are derived. On the other hand, we obtain some information about the elements of the matrix A from this asymptotic behaviour.

2. Representation Formulas and Basic Tools

We need to introduce the kernel matrices

$$\mathbb{K}_n(x, c) = \begin{pmatrix} K_n(x, c) \\ K_n^{(0,1)}(x, c) \\ \vdots \\ K_n^{(0,r)}(x, c) \end{pmatrix}$$

$$\mathbb{K}_n = \begin{pmatrix} K_n(c,c) & K_n^{(0,1)}(c,c) & \cdots & K_n^{(0,r)}(c,c) \\ K_n^{(1,0)}(c,c) & K_n^{(1,1)}(c,c) & \cdots & K_n^{(1,r)}(c,c) \\ \vdots & \vdots & \ddots & \vdots \\ K_n^{(r,0)}(c,c) & K_n^{(r,1)}(c,c) & \cdots & K_n^{(r,r)}(c,c) \end{pmatrix}$$

where $K_n(x, y) = \sum_{j=0}^{n} P_j(x)\, P_j(y)/\|P_j\|_\mu^2$ is the kernel polynomial associated with the orthogonal polynomials (P_n) and $K_n^{(i,j)}(x, y) = (\partial^{i+j}/\partial x^i \partial y^j)\, K_n(x, y)$. Note that the $(r+1)\times(r+1)$ matrix $\mathbb{H}_n$ is symmetric and positive definite.

The following identity turns out to be convenient (see [7, § 0.8.5 on p. 21–22])

LEMMA 1. *Let* $x,\ u \in \mathbf{R}^k$, $x_0 \in \mathbf{R}$ *and* $A \in \mathbf{R}^{k\times k}$. *Then*

$$\begin{vmatrix} x_0 & x' \\ u & A \end{vmatrix} = (\det A)(x_0 - x'A^{-1}u)$$

From it, we obtain formula (2.2) that will be relevant in Section 4.

PROPOSITION 2. *For every* $n \in \mathbf{N}$

(i) *The matrix* $I + A\mathbb{H}_{n-1}$ *is non singular*

(ii) $$Q_n(x) = P_n(x) - \mathbb{P}_n(c)'(I + A\mathbb{H}_{n-1})^{-1} A\mathbb{K}_{n-1}(x, c) \tag{2.1}$$

(iii) $$(Q_n, Q_n)/\|P_n\|_\mu^2 = \det(I + A\mathbb{H}_n)/\det(I + A\mathbb{H}_{n-1}) \tag{2.2}$$

Proof. (i) As the matrices A and $\mathbb{H}_{n-1}^{-1}$ are positive semidefinite and definite respectively, the matrix $\mathbb{H}_{n-1}^{-1} + A$ is non singular. So $I + A\mathbb{H}_{n-1} = [\mathbb{H}_{n-1}^{-1} + A]\, \mathbb{H}_{n-1}$ is also non singular.

(ii) Consider the Fourier expansion of Q_n in terms of (P_n), then

$$Q_n(x) = P_n(x) - \mathbb{Q}_n(c)'\, A\mathbb{K}_{n-1}(x, c) \tag{2.3}$$

If we evaluate the polynomial $Q_n(x)$ and their successive derivatives at the point c, from i) we get $\mathbb{Q}_n(c)' = \mathbb{P}_n(c)'(I + A\mathbb{H}_{n-1})^{-1}$ and because of (2.3), the formula (2.1) is true.

(iii) From orthogonality and the above expression for $\mathbb{Q}_n(c)'$ we have

$$\begin{aligned}(Q_n, Q_n) = (Q_n, P_n) &= \int_I Q_n P_n\, d\mu + \mathbb{Q}_n(c)'\, A\mathbb{P}_n(c) \\ &= \|P_n\|_\mu^2 + \mathbb{P}_n(c)'(I + A\mathbb{H}_{n-1})^{-1} A\mathbb{P}_n(c) \end{aligned} \tag{2.4}$$

On the other hand, from Lemma 1 and by straightforward calculations with determinants, formula (2.1) can be written as

$$Q_n(x) = (\det[I + A\mathbb{H}_{n-1}])^{-1} \begin{vmatrix} P_n(x) & \mathbb{P}_n(c)' \\ A\mathbb{K}_n(x, c) & I + A\mathbb{H}_n \end{vmatrix} \tag{2.5}$$

Besides, from (2.5) and Lemma 1, we have

$$\|P_n\|_\mu^2 = \int_I Q_n P_n d\mu = \begin{vmatrix} \|P_n\|_\mu^2 & \mathbb{P}_n(c)' \\ A\mathbb{P}_n(c) & I + A\mathbb{H}_n \end{vmatrix} (\det[I + A\mathbb{H}_{n-1}])^{-1}$$

$$= \frac{\det(I + A\mathbb{H}_n)}{\det(I + A\mathbb{H}_{n-1}} \|P_n\|_\mu^2 + \frac{\begin{vmatrix} 0 & \mathbb{P}_n(c)' \\ A\mathbb{P}_n(c) & I + A\mathbb{H}_{n-1} \end{vmatrix}}{\det(I + A\mathbb{H}_{n-1})}$$

$$= \frac{\det(I + A\mathbb{H}_n)}{\det(I + A\mathbb{H}_{n-1})} \|P_n\|_\mu^2 - \mathbb{P}_n(c)'(I + A\mathbb{H}_{n-1})^{-1} A\mathbb{P}_n(c).$$

Thus, from (2.4), the result follows. ∎

It is well known that the kernels satisfy the Christoffel–Darboux formula, (see p. 23 in [4])

$$K_{n-1}(x, y) = \frac{1}{\|P_{n-1}\|_\mu^2} \frac{P_n(x) P_{n-1}(y) - P_n(y) P_{n-1}(x)}{x - y}$$

If we take successive derivatives with respect to y and evaluate at $y = c$, by using Leibniz's formula, we find that, for $j = 0, ..., r$

$$K_{n-1}^{(0,j)}(x, y) = \frac{j!}{\|P_{n-1}\|_\mu^2} \frac{1}{(x - c)^{j+1}} [T_j(x, c; P_{n-1}) P_n(x) - T_j(x, c; P_n) P_{n-1}(x)]$$

where $T_j(x, c; f)$ is the $j-th$ Taylor polynomial of f in c. Inserting this into (2.3) and denoting $B(x) = 1/\|P_{n-1}\|_\mu^2 [I + A\mathbb{H}_{n-1}]^{-1} A(x-c)^r D(x)$ where

$$D(x) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1!(x-c)^{-1} & 1 & 0 & \cdots & 0 \\ 2!(x-c)^{-2} & \frac{2!}{1!}(x-c)^{-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r!(x-c)^{-r} & \frac{r!}{1!}(x-c)^{-(r-1)} & \frac{r!}{2!}(x-c)^{-(r-2)} & \cdots & 1 \end{pmatrix}$$

we obtain:

PROPOSITION 3. *The formula*

$$(x-c)^{r+1} Q_n(x) = q_{r+1}(x, n) P_n(x) + q_r(x, n) P_{n-1}(x) \tag{2.6}$$

where $q_{r+1}(x, n) = (x-c)^{r+1} - \mathbb{P}_n(c)^t B(x) \mathbb{P}_{n-1}(c)$ *and* $q_r(x, n) = \mathbb{P}_n(c)^t B(x) \mathbb{P}_n(c)$ *holds.*

We point out that $B(x)$ is a matrix polynomial of degree precisely r and coefficients depending on n, hence $\deg q_{r+1} = r+1$ and $\deg q_r = r$. So, from (2.6) it follows that the sequence (Q_n) is strictly quasi-orthogonal of order $r+1$ with respect to the measure $(x-c)^{r+1} d\mu$ and therefore $(x-c)^{r+1} Q_n(x) = P_{n+r+1}(x) + \sum_{j=n-r-1}^{n+r} \lambda_{nj} P_j(x)$ where $\lambda_{n,n-r-1} \neq 0$. The numbers λ_{nj} can be expressed in terms of the coefficients of the polynomials $q_{r+1}(x, n)$ and $q_r(x, n)$ and the coefficients of the three term recurrence relation satisfied by the SMOP (P_n).

Concerning the general properties about zeros of the polynomials Q_n, the following result can be derived:

PROPOSITION 4. *The polynomial Q_n has at least $n-(r+1)$ different zeros with odd multiplicity in $\mathring{I}$ when r is odd and $n \geqslant r+2$ and at least $n-(r+2)$ when r is even and $n \geqslant r+3$. Moreover, if $c \notin \mathring{I}$, Q_n has at least $n-(r+1)$ different zeros with odd multiplicity in $\mathring{I}$ for $n \geqslant r+2$.*

Next, we consider the particular case where A is a 2×2 matrix. We will denote it by $A = \begin{pmatrix} M & \lambda \\ \lambda & N \end{pmatrix}$. Since A is positive semidefinite, we have $M \geqslant 0$ and $\lambda^2 \leqslant MN$ (therefore $N \geqslant 0$). In this situation, $\mathbb{H}_{n-1}$ is also a 2×2 matrix and $\det \mathbb{H}_{n-1} = K_{n-1}(c, c) K_{n-1}^{(1,1)}(c, c) - (K_{n-1}^{(0,1)}(c, c))^2$.

We will denote by Q_n^λ the monic polynomials orthogonal with respect to the inner product

$$(f, g)_\lambda = \int_I fg \, d\mu + (f(c), f'(c)) \begin{pmatrix} M & \lambda \\ \lambda & N \end{pmatrix} \begin{pmatrix} g(c) \\ g'(c) \end{pmatrix} \tag{2.7}$$

Note that when $\lambda = 0$ we recover the diagonal case and then $Q_n^0(x) = Q_n(x)$.

Let $(P_n(x; \mu_2))$ and $(P_n(x; \mu_4))$ be the SMOP with respect to the measures $d\mu_2 = (x-c)^2 d\mu$ and $d\mu_4 = (x-c)^4 d\mu$, respectively and $(K_n(x, y; \mu_2))$ the sequence of kernels associated with $(P_n(x; \mu_2))$.

We will use several expressions concerning the diagonal case (see formulas (2.3) and (2.10) in [2]) that we next summarize.

LEMMA 5. *Let $\lambda = 0$, then*

(i) $$\begin{aligned} \Delta_n(0) &= \det(I + A\mathbb{H}_n) \\ &= 1 + MK_{n-1}(c, c) + NK_{n-1}^{(1,1)}(c, c) + MN \det \mathbb{H}_{n-1} \end{aligned}$$

(ii) $\Delta_n(0)\, Q_n(c) = P_n(c)[1 + NK_{n-1}^{(1,1)}(c, c)] - P_n'(c)\, NK_{n-1}^{(0,1)}(c, c)$

$\Delta_n(0)\, Q_n'(c) = -P_n(c)\, MK_{n-1}^{(0,1)}(c, c) + P_n'(c)[1 + MK_{n-1}(c, c)]$

(iii) *Suppose that the condition $P_n(c)\, P_{n-1}(c; \mu_2) \neq 0$ is satisfied for every $n \in \mathbf{N}$. The formula*

$$Q_n(x) = (1 - \alpha_n)\, P_n(x) + (\alpha_n - \beta_n)(x - c)\, P_{n-1}(x; \mu_2) + \beta_n (x - c)^2\, P_{n-2}(x; \mu_4)$$

where $\alpha_n = 1 - P_n(c)^{-1}\, Q_n(c)$ and $\beta_n = NP_{n-1}(c; \mu_2)^{-1}\, Q_n'(c)\, K_{n-2}(c, c; \mu_2)$, holds.

Now, we can obtain several expressions for the polynomials Q_n^λ that will be essential in sections 3 and 4.

PROPOSITION 6. *We have*

(a) $$\Delta_n(\lambda) = \det(I + A\mathbb{H}_n) = \Delta_n(0) + 2\lambda K_{n-1}^{(0,1)}(c, c) - \lambda^2 \det \mathbb{H}_{n-1} \tag{2.8}$$

(b)
$$\begin{aligned} Q_n^\lambda(x) = P_n(x) & - \frac{K_{n-1}(x, c)}{\Delta_n(\lambda)} [MP_n(c) + \lambda P_n'(c) + (MN - \lambda^2) \\ & \times (P_n(c)\, K_{n-1}^{(1,1)}(c, c) - P_n'(c)\, K_{n-1}^{(0,1)}(c, c))] \\ & - \frac{K_{n-1}^{(0,1)}(x, c)}{\Delta_n(\lambda)} [NP_n'(c) + \lambda P_n(c) + (MN - \lambda^2) \\ & \times (P_n'(c)\, K_{n-1}(c, c) - P_n(c)\, K_{n-1}^{(0,1)}(c, c))] \end{aligned} \tag{2.9}$$

(c) $$Q_n^\lambda(c) = \Delta_n(\lambda)^{-1}[\Delta_n(0)\, Q_n(c) - \lambda P_{n-1}(c; \mu_2)\, K_{n-1}(c, c)] \tag{2.10}$$

(d) $$Q_n^{\lambda'}(c) = \Delta_n(\lambda)^{-1}[\Delta_n(0)\, Q_n'(c) + \lambda P_{n-1}'(c; \mu_2)\, K_{n-1}(c, c)] \tag{2.11}$$

Proof. (a) By a simple calculation we get,

$$\Delta_n(\lambda) = 1 + MK_{n-1}(c, c) + NK_{n-1}^{(1,1)}(c, c) + 2\lambda K_{n-1}^{(0,1)}(c, c) + (MN - \lambda^2) \det \mathbb{H}_{n-1} \tag{2.12}$$

Then, from Lemma 5-i), (2.8) follows. Note that, from Proposition 2, $\Delta_n(\lambda) \neq 0$.

(b) It is a direct consequence of the formula (2.1).

(c) From b) and Lemma 5-ii), we get that

$$\Delta_n(\lambda)\, Q_n^\lambda(c) - \Delta_n(0)\, Q_n(c) = \lambda[\, P_n(c)\, K_{n-1}^{(0,1)}(c, c) - P_n'(c)\, K_{n-1}(c, c)]$$

Taking into account Lemma 2.1 in [2], the formula for $Q_n^\lambda(c)$ follows.

(d) By differentiating in (2.9) and evaluating at $x = c$ we obtain $\Delta_n(\lambda)\, Q_n^{\lambda\prime}(c)$. Now it suffices to proceed as in c).

We can write $K_{n-1}(x, c)$ and $K_{n-1}^{(0,1)}(x, c)$ in terms of $P_n(x)$, $P_{n-1}(x; \mu_2)$ and $P_{n-2}(x; \mu_4)$. By substitution of these values in (2.9) we get a representation formula that will be a basic tool to obtain asymptotic results:

PROPOSITION 7. *Let c be such that the condition $P_n(c)\, P_{n-1}(c; \mu_2) \neq 0$ is satisfied for every $n \in \mathbf{N}$. Then, the formula*

$$\begin{aligned} Q_n^\lambda(x) = (1 - \alpha_n^\lambda)\, P_n(x) + (\alpha_n^\lambda - \beta_n^\lambda)(x - c)\, P_{n-1}(x; \mu_2) \\ + \beta_n^\lambda (x - c)^2\, P_{n-2}(x; \mu_4) \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \alpha_n^\lambda = \frac{\Delta_n(0)}{\Delta_n(\lambda)}\, \alpha_n + \frac{\lambda}{\Delta_n(\lambda)\, P_n(c)} \\ \times [-\lambda P_n(c) \det \mathbb{H}_{n-1} + P_n(c)\, K_{n-1}^{(0,1)}(c, c) + P_n'(c)\, K_{n-1}(c, c)] \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \beta_n^\lambda = \frac{\Delta_n(0)}{\Delta_n(\lambda)}\, \beta_n + \frac{\lambda}{\Delta_n(\lambda)}\, \frac{K_{n-2}(c, c; \mu_2)}{P_{n-1}(c; \mu_2)} \\ \times [P_n(c) - \lambda P_{n-1}(c; \mu_2)\, K_{n-1}(c, c)] \end{aligned} \tag{2.15}$$

holds.

Note that if c is off the support of μ then for every n, $P_n(c)\, P_{n-1}(c; \mu_2) \neq 0$ holds.

3. ZEROS IN THE CASE 2×2

Because of proposition 4, we know that for $c \in \mathbf{R}$ and $n \geqslant 3$ the polynomial Q_n^λ has at least $n - 2$ different zeros with odd multiplicity in $\mathring{I}$, namely $\xi_{n1} \leqslant \cdots \leqslant \xi_{nk}$ with $k \geqslant n - 2$. With additional assumptions, we will improve this result.

3.1. In the diagonal case, if $c \notin \mathring{I}$ the zeros are real, simple and at least $n-1$ of them belong to either the interval $(\inf I, c)$ or $(c, \sup I)$, according to $c \geqslant \sup I$ or $c \leqslant \inf I$, (see [18]). Now we are going to investigate these properties for the polynomials associated with the inner product (2.7) when $c \notin \mathring{I}$. For the sake of simplicity we will only work with an interval I with $\sup I < \infty$ and $c \geqslant \sup I$, the results for an interval I with $\inf I > -\infty$ and $c \leqslant \inf I$ being analogous.

PROPOSITION 8. *If* $0 \leqslant \lambda \leqslant \sqrt{MN}$ *with* $MN > 0$, *then for* $n \geqslant 3$ *the polynomial* Q_n^λ *has real and simple zeros; at least* $n-1$ *of them are located in* $(\inf I, c)$. *Moreover, if* Q_n^λ *has a zero* ξ_{nn} *outside of* $(\inf I, c)$, *then* $c \leqslant \xi_{nn} < c + (c - \xi_{n1})/(n-1)$ *and* $\xi_{nn} - c < c - \xi_{n,n-1}$.

Proof. Since $c \geqslant \sup I$, Lemmas 2.1 in [2] and 5-ii) imply $Q_n'(c) > 0$ for all $n \in \mathbf{N}$, then $Q_n^{\lambda\prime}(c) > 0$ follows from proposition 6-d). We write $p(x) = (x - \xi_{n1}) \cdots (x - \xi_{nk})$.

If $k = n-2 (n \geqslant 3)$, then $(Q_n^\lambda(x), (x-c)\, p(x)) = 0 = (Q_n^\lambda(x), p(x))$. From these conditions it follows that $Q_n^\lambda(c)\, Q_n^{\lambda\prime}(c) \neq 0$ and

$$\operatorname{sgn}[\lambda Q_n^\lambda(c) + N Q_n^{\lambda\prime}(c)] = -\operatorname{sgn}\{[M Q_n^\lambda(c) + \lambda Q_n^{\lambda\prime}(c)] + [\lambda Q_n^\lambda(c) + N Q_n^{\lambda\prime}(c)]\}$$

Thus, sgn $[\lambda Q_n^\lambda(c) + N Q_n^{\lambda\prime}(c)] = -\operatorname{sgn}[M Q_n^\lambda(c) + \lambda Q_n^{\lambda\prime}(c)]$ holds, and therefore we have $Q_n^\lambda(c)\, Q_n^{\lambda\prime}(c) < 0$. Since $Q_n^{\lambda\prime}(c) > 0$, we get $Q_n^\lambda < 0$ which implies $\xi_{nn} > c$. Besides, taking into account that

$$(Q_n^\lambda, (x-c)^2 p) = \int_I (x-c)^2\, Q_n^\lambda\, p\, d\mu > 0$$

we deduce that Q_n^λ has not a zero in $(-\infty, \inf I]$.

If $k = n-1$, obviously all the roots are real and simple. Suppose that ξ_{nn} is outside of the interval $(\inf I, c)$; by orthogonality $(Q_n^\lambda, p) = 0$ and since the sign of $Q_n^\lambda(x)\, p(x)$ does not change in $(\inf I, c)$ we obtain that $Q_n^\lambda(c) \leqslant 0$ and so $\xi_{nn} \geqslant c$.

In addition, if $\xi_{nn} > c$, from $(Q_n'(c)/Q_n(c)) = \sum_{j=1}^{n-1} 1/(c - \xi_{nj}) - 1/(\xi_{nn} - c) < 0$ we have that $\xi_{nn} - c < c - \xi_{n1}/n-1$ and $\xi_{nn} - c < c - \xi_{n,n-1}$.

In general, for any $\lambda \neq 0$, we cannot assure that the above result remains true for all $n \geqslant 3$; however, with some additional assumptions, a similar result holds for n large enough:

PROPOSITION 9. *Let us suppose* I *bounded and* $c > \sup I$, *then for all* $\lambda \in [-\sqrt{MN}, +\sqrt{MN}]$ *with* $N > 0$ *there exists a positive integer* n_0, *such that for* $n \geqslant n_0$, *the polynomial* Q_n^λ *has only real and simple zeros;* $n-1$ *of them are located in* $(\inf I, c)$ *and the largest zero is greater than* c.

Proof. From formula (2.10), taking into account Lemmas 2.1 in [2] and 5(ii), we have that $Q_n^\lambda(c) < 0$ if and only if

$$P_n(c) - NP'_{n-1}(c; \mu_2)\, K_{n-1}(c, c) < \lambda P_{n-1}(c; \mu_2)\, K_{n-1}(c, c)$$

so, it suffices to show that

$$\lim_{n\to\infty}\left[\frac{P_n(c)}{P_{n-1}(c;\mu_2)\, K_{n-1}(c,c)} - N\frac{P'_{n-1}(c;\mu_2)}{P_{n-1}(c;\mu_2)}\right] = -\infty$$

Indeed, let us write $a = \inf I$ and denote by $x_{n1} < x_{n2} < \cdots < x_{nn}$ the zeros of P_n. Since $c \geqslant \sup I$ we have $(P'_n(c)/P_n(c)) = \sum_{k=1}^n 1/(c - x_{nk}) > n/(c - x_{n1}) > n/(c-a)$ and so $\lim_{n\to\infty} (P'_n(c)/P_n(c)) = +\infty$. On the other hand from the interlacing property, i.e. between two consecutive zeros of $P_n(x)$ there is exactly one zero of $P_{n-1}(x; \mu_2)$ (see [16, Lemma 6.1]), $\lim_{n\to\infty}(P'_{n-1}(c;\mu_2)/P_{n-1}(c;\mu_2)) = +\infty$ follows.

Finally, it can be easily proved that the sequence $(P'_n(c)/P_{n-1}(c;\mu_2)\, K_{n-1}(c,c))$ is decreasing with positive terms, see Lemma 4.3 in [13]; hence

$$\lim_{n\to\infty}\frac{P_n(c)}{P_{n-1}(c;\mu_2)\, K_{n-1}(c,c)} = \lim_{n\to\infty}\frac{P'_n(c)}{P_{n-1}(c;\mu_2)\, K_{n-1}(c,c)}\frac{P_n(c)}{P'_n(c)} = 0$$

Hence, there exists a positive integer n_0 such that $Q_n(c) < 0$ for $n \geqslant n_0$. Now, proceeding as in Proposition 8, the result follows.

Remark. In the above proposition, the boundedness of I has only been used in order to assure that the conditions $\lim_{n\to\infty} (P'_n(c)/P_n(c)) = \lim_{n\to\infty}(P'_{n-1}(c;\mu_2)/P_{n-1}(c;\mu_2)) = +\infty$ hold. So, the proposition remains valid whenever both conditions are true.

Next, if $\xi_{nn} > c$, we can estimate the distance from ξ_{nn} to c in terms of the parameters M, N and λ.

PROPOSITION 10. *Let $\lambda > 0$ and $\xi_{nn} > c$. The following assertions are true*:

(i) *If $MN \geqslant 2\lambda^2$, then $\xi_{nn} - c < N/2\sqrt{MN - \lambda^2}$.*

(ii) *If $\lambda^2 \leqslant MN \leqslant 2\lambda^2$, then $\xi_{nn} - c < \lambda/M$.*

Proof. The polynomial Q_n^λ can be written as $Q_n^\lambda(x) = (\xi_{nn} - x)\,\phi(x)$ where $\phi(c) < 0$ and $\phi'(c) < 0$. By orthogonality $(Q_n^\lambda, \phi) = 0$ and since $\int_I (\xi_{nn} - x)\,\phi^2(x)\, d\mu(x) > 0$ we obtain

$$(\xi_{nn} - c)[M\phi^2(c) + 2\lambda\phi(c)\,\phi'(c) + N(\phi')^2(c)] < \lambda\phi^2(c) + N\phi(c)\,\phi'(c)$$

It is not difficult to prove that the expression inside the brackets is positive for all $\lambda \in \mathbf{R}$. Hence

$$\xi_{nn} - c < \frac{\lambda + N(\phi'(c)/\phi(c))}{M + 2\lambda(\phi'(c)/\phi(c)) + N(\phi'(c)/\phi(c))^2}$$

Therefore, it suffices to find the maximum value of $(\lambda + Nx)/(M + 2\lambda x + Nx^2)$ in $(0, +\infty)$ (note that $(\phi'(c)/\phi(c)) > 0$) or equivalently the maximum value of $y/(a + x^2)$ in $(\lambda/N, +\infty)$ where $a = (MN - \lambda^2)/N^2$. By elementary calculations the result follows. ∎

Remark. Note that i) is also true for $\lambda = 0$ and in this case we recover the estimates obtained in [2], [10] and [16]. On the other hand, whenever $MN > \lambda^2$ and $\lambda < 0$ the same argument implies $\xi_{nn} - c < N/\sqrt{2(MN - \lambda^2)}$.

3.2. In the diagonal case, when μ is symmetric (i.e. invariant under the transformation $x \to -x$ and I a symmetric interval) if we take $c = 0$, a special situation appears: all the zeros of Q_n are real, simple, belong to $\mathring{I}$ and they are symmetric with respect to 0 (see [2]). We want to point out that the above assumptions concerning I and μ imply that $K_n^{(0,1)}(0, 0) = 0$ for all $n \in \mathbf{N}$.

We are going to analyze the analogous problem in the non-diagonal case. Firstly, observe that Q_n^λ are symmetric (i.e. $Q_n^\lambda(-x) = (-1)^n Q_n^\lambda(x)$) if and only if $\lambda = 0$. We assume that μ is symmetric, $c = 0$ and A is singular, that is $MN = \lambda^2$. Under these assumptions, by using the Christoffel–Darboux formula, (2.9) becomes

$$x^2 Q_{2n}^\lambda(x) = q_2^\lambda(x, 2n)\, P_{2n}(x) + q_1^\lambda(x, 2n)\, P_{2n-1}(x) \tag{3.1}$$

where

$$q_2^\lambda(x, 2n) = x^2 - \lambda \Delta_{2n}(\lambda)^{-1} \|P_{2n-1}\|^{-2} P_{2n}(0)\, P'_{2n-1}(0)\, x$$
$$q_1^\lambda(x, 2n) = \Delta_{2n}(\lambda)^{-1} \|P_{2n-1}\|^{-2} P_{2n}^2(0)[Mx + \lambda]$$

and

$$x^2 Q_{2n+1}^\lambda(x) = r_2^\lambda(x, 2n+1)\, P_{2n+1}(x) + r_1^\lambda(x, 2n+1)\, P_{2n}(x) \tag{3.2}$$

where

$$r_2^\lambda(x, 2n+1) = x^2 - \Delta_{2n+1}(\lambda)^{-1} \|P_{2n}\|^{-2} P'_{2n+1}(0)\, P_{2n}(0)[\lambda x + N]$$
$$r_1^\lambda(x, 2n+1) = N\Delta_{2n+1}(\lambda)^{-1} \|P_{2n}\|^{-2} P'^{2}_{2n+1}(0)\, x$$

We denote by $(x_{2n,k})_1^{2n}$, $(y_{2n-1,k})_1^{2n-1}$ the zeros of the polynomials P_{2n} and P_{2n-1} respectively, ordered by increasing size.

PROPOSITION 11. *Suppose μ symmetric, $c=0$ and $MN=\lambda^2$. The zeros of the polynomial Q_{2n}^{λ} are real and simple and they belong to the interval* $[\min\{x_0, x_{2n,1}\}, \max\{x_0, x_{2n,2n}\}]$ *where $x_0=-\lambda M^{-1}$. Moreover, at least one zero of Q_{2n}^{λ} can be found between two consecutive zeros of P_{2n}.*

Proof. By using the Christoffel–Darboux formula and taking into account the value of $\Delta_{2n}(\lambda)$ we can write the polynomial q_2^{λ} in (3.1) as

$$q_2^{\lambda}(x, 2n)=\Delta_{2n}(\lambda)^{-1}\, x[(1+NK_{2n-1}^{(1,1)}(0,0))\, x+(Mx+\lambda)\, K_{2n-1}(0,0)]$$

and thus $q_2^{\lambda}(x, 2n)>0$ for every $x\in(-\infty, \min\{x_0, x_{2n,1}\})\cup(\max\{x_0, x_{2n,2n}\}, +\infty)$. Since $q_1^{\lambda}(x_0, 2n)=0$, we can easily deduce that

$$Q_{2n}^{\lambda}(x)>0 \quad \text{for all} \quad x\in(-\infty, \min\{x_0, x_{2n,1}\})\cup(\max\{x_0, x_{2n,2n}\}, +\infty) \tag{3.3}$$

If $x_0\notin[x_{2n,k}, x_{2n,k+1}]$ for some $k\in\{1, ..., 2n-1\}$, as the sign of $q_1^{\lambda}(x, 2n)$ is constant in the interval $[x_{2n,k}, x_{2n,k+1}]$ then, from (3.1) and the separation property of the zeros of the polynomials P_n, the polynomial Q_{2n}^{λ} changes its sign in $x_{2n,k}$ and $x_{2n,k+1}$.

On the other hand, it follows from (3.1) that P_{2n} and Q_{2n}^{λ} have at most one common zero, namely x_0. We will distinguish two cases according to whether x_0 is or isn't a zero of P_{2n}.

If $P_{2n}(x_0)\neq 0$, we have the following situations: (a) $x_0<x_{2n,1}$; (b) $x_0>x_{2n,2n}$, and (c) there exists a unique $k\in\{1, ..., 2n-2\}$ such that $x_0\in(x_{2n,k}, x_{2n,k+1})$.

(a) If $x_0<x_{2n,1}$, we have $x_0\notin[x_{2n,k}, x_{2n,k+1}]$ for every $k=1, ..., 2n-1$ and then the polynomial Q_{2n}^{λ} has only one zero between two consecutive zeros of P_{2n}. Therefore, all the zeros are real, simple and from (3.3) we obtain

$$x_0<\xi_{2n,1}<x_{2n,1}<\xi_{2n,2}<\cdots<\xi_{2n,2n}<x_{2n,2n}$$

In the case b), the result can be proved by the same arguments and we get

$$x_{2n,1}<\xi_{2n,1}<x_{2n,2}<\cdots<x_{2n,2n}<\xi_{2n,2n}<x_0$$

(c) We can assure that Q_{2n}^{λ} has at least one zero in each interval $(x_{2n,j}, x_{2n,j+1})$ with $j\neq k$. Moreover, if we compare the signs of the polynomial Q_{2n}^{λ} in $x_{2n,k}$, x_0 and $x_{2n,k+1}$ we obtain that Q_{2n}^{λ} has a zero in $(x_{2n,k}, x_0)$ and another in $(x_0, x_{2n,k+1})$.

Now, we suppose that $P_{2n}(x_0)=0$. First, let $x_0=x_{2n,k}$ where $k\neq 0, 2n$. Then there is at least a zero of Q^{λ}_{2n} in $(x_{2n,j}, x_{2n,j+1})$ for $j=1, ..., 2n-1$ with $j\neq k-1$, k. Let us assume $k\leqslant n$, that is $x_0<0$. Since

$$\operatorname{sgn} Q^{\lambda}_{2n}(x_{2n,k-1}) = -\operatorname{sgn} P_{2n-1}(x_{2n,k-1}) = (-1)^k$$

and

$$\operatorname{sgn} Q^{\lambda}_{2n}(y_{2n-1,k-1}) = \operatorname{sgn} P_{2n}(y_{2n-1,k-1}) = (-1)^{k-1}$$

Q^{λ}_{2n} has a zero in $(x_{2n,k-1}, y_{2n-1,k-1})$. Moreover, as obviously $q^{\lambda}_2(x_0, 2n)>0$, we can take y such that $x_0<y<y_{2n-1,k}$ and $q^{\lambda}_2(y, 2n)>0$. Applying once again (3.1) we get that Q^{λ}_{2n} changes its sign in y and $x_{2n,k+1}$; therefore Q^{λ}_{2n} has also a zero in $(x_0, x_{2n,k+1})$ which proves the result. In the case $k\geqslant n+1$, the same arguments are valid.

Finally, in a similar way as before, the proposition can be proved for $x_0=x_{2n,1}$ and for $x_0=x_{2n,2n}$.

We want to note that in this case, although the polynomials Q^{λ}_{2n} are not symmetric, they have n positive and n negative zeros as it happens for the symmetric polynomials. For this, it suffices to locate the zeros of Q^{λ}_{2n} in $(x_{2n,n}, x_{2n,n+1})$ by applying adequately the formula (3.1) and taking into account that in this case, (2.9) implies that $\operatorname{sgn} Q_{2n}(0)=\operatorname{sgn} P_{2n}(0)$.

PROPOSITION 12. *Suppose μ symmetric, $c=0$ and $MN=\lambda^2$. The polynomial Q^{λ}_{2n+1} has at least $2n-1$ real and simple zeros which belong to $\mathring{I}$. Moreover, if $\lambda>0$ (respectively $\lambda<0$) there are at least n negative (respectively positive) zeros and $n-1$ positive (respectively negative) zeros; each one of them is located between two consecutive zeros $y_{2n+1,j}$ and $y_{2n+1,j+1}$ of P_{2n+1} for $j=1, ..., 2n$, $j\neq n+1$ (respectively $j\neq n$).*

Proof. We know, by Proposition 4, that Q^{λ}_{2n+1} has at least $2n-1$ different zeros with odd multiplicity in $\mathring{I}$.

From (3.2) and the separation property of the zeros of the polynomials P_n, since $\operatorname{sgn} r^{\lambda}_1(x, 2n+1)=\operatorname{sgn} x$, we can say that Q^{λ}_{2n+1} has at least one zero in every interval $(y_{2n+1,j}, y_{2n+1,j+1})$ whenever $j=1, ..., 2n$ with $j\neq n$ and $j\neq n+1$ and also in $(y_{2n+1,n}, y_{2n+1,n+2})$ because of

$$\operatorname{sgn} Q^{\lambda}_{2n+1}(y_{2n+1,n}) = -\operatorname{sgn} P_{2n}(y_{2n+1,n}) = (-1)^n$$

$$\operatorname{sgn} Q^{\lambda}_{2n+1}(y_{2n+1,n+2}) = -\operatorname{sgn} P_{2n}(y_{2n+1,n+2}) = (-1)^{n+1}$$

Now, from (2.9), it follows that $\operatorname{sgn} Q^{\lambda}_{2n+1}(0)=(-1)^{n+1}\operatorname{sgn}\lambda$ and the proof is complete. ∎

We observe, with aid of Mathematica, that for $A = \begin{pmatrix} \lambda & \lambda \\ \lambda & \lambda \end{pmatrix}$, μ the Legendre measure and $c = 0$, there exists $\lambda_0 \in (0, +\infty)$ such that the polynomials Q_n^λ have two complex zeros for $n = 3, 5$ whenever $\lambda > \lambda_0$. The existence of complex zeros for Q_{2n+1}^λ remains an open problem.

Concerning the separation property of the zeros of the polynomials P_n and Q_n^λ when the matrix A is singular note that, for a symmetric measure μ and $c = 0$, a result has just been obtained in propositions 11 and 12. Now we study the same property when $c \notin \mathring{I}$ and μ is not necessarily symmetric.

PROPOSITION 13. *Let $\lambda \geq 0$, $c \geq \sup I$ and suppose $MN = \lambda^2$. Then, between two consecutive zeros of P_n there is a unique zero of Q_n^λ. Moreover the largest zero of Q_n^λ is greater than the one of P_n.*

Proof. From (2.9), we have

$$Q_n^\lambda(x) - P_n(x) = \frac{-1}{\Delta_n(\lambda)} ([MP_n(c) + \lambda P_n'(c)]\, K_{n-1}(x, c) + [NP_n'(c) + \lambda P_n(c)]\, K_{n-1}^{(0,1)}(x, c)) \tag{3.4}$$

On the other hand, by using the Christoffel–Darboux formula, the polynomial $K_{n-1}(x, c)$ has real and simple zeros which interlace with the zeros of P_n. The same result is true for $K_{n-1}^{(0,1)}(x, c)$ (see Lemma 5.1 in [13]). Then if $(x_{ni})_{i=1}^n$ are the zeros of P_n, from (3.4), the polynomial $Q_n^\lambda(x)$ changes its sign at each x_{ni}, $i = 1, ..., n$. Moreover, from (3.4), $Q_n^\lambda(x_{nn}) < 0$ and thus $\xi_{nn} > x_{nn}$.

Remark. Proposition 13 implies that Q_n^λ has at least $n - 1$ zeros in $\mathring{I}$, which improves, for A singular, the result in Proposition 8.

4. RELATIVE ASYMPTOTICS FOR c OFF $\operatorname{supp} \mu$ IN THE CASE 2×2

In this section we are going to analyze the asymptotic behaviour of the Sobolev-type orthonormal polynomials with respect to the standard orthonormal polynomials and to compare the results with the previous ones obtained by López, Marcellán and Van Assche (see [11] and [15]). In order to do this we will assume that the measure μ belongs to the Nevai class $M(0, 1)$. Recall that this means that $\operatorname{supp} \mu = [-1, 1] \cup E$ with E a set which is at most denumerable and $E' \subset \{-1, 1\}$.

Let (p_n) be the sequence of orthonormal polynomials with respect to such a measure μ and define $\varphi(x) = x + \sqrt{x^2 - 1}$ where the square root is such that $|\varphi(x)| > 1$ whenever $x \in \mathbb{C} \backslash [-1, 1]$, which implies that

$\sqrt{x^2-1}>0$ for $x>1$ and $\sqrt{x^2-1}<0$ for $x<-1$. It is known that for all $j=0, 1, 2, \ldots$

$$\lim_n \frac{p_n^{(j)}(x)}{p_{n-1}^{(j)}(x)} = \varphi(x) \tag{4.1}$$

$$\lim_n \frac{p_n^{(j+1)}(x)}{n p_n^{(j)}(x)} = \frac{1}{\sqrt{x^2-1}} \tag{4.2}$$

uniformly on compact subsets of $\mathbf{C}\backslash\operatorname{supp}\mu$ (see [11] and p. 33–36 in [17]).

Moreover, $\mu \in M(0, 1)$ implies that $\mu_2 \in M(0, 1)$ (recall that $d\mu_2 = (x-c)^2\, d\mu$) and as a consequence we have (see [17], Theorems 20, 26 and 29, p. 68–73)

$$\lim_n \frac{p_n(x; \mu_2)}{p_n(x)} = \frac{1}{2\,|\varphi(c)|} \left(\frac{\varphi(x)-\varphi(c)}{x-c} \right)^2 \tag{4.3}$$

uniformly on compact subsets of $(\mathbf{C}\backslash\operatorname{supp}\mu)\backslash\{c\}$,

$$\lim_n \frac{p_n(c; \mu_2)}{p_n(c)} = \frac{\varphi'^2(c)}{2\,|\varphi(c)|} \tag{4.4}$$

and

$$\lim_n \frac{\|P_{n-1}(.; \mu_2)\|_{\mu_2}}{\|P_n\|_\mu} = |\varphi(c)| \tag{4.5}$$

Before we study the relative asymptotics we need to known the "size" of the kernels relative to the polynomials p_n and their derivatives. The following discrete version of l'Hospital's rule is often very helpful in order to calculate limits (see e.g. [8])

Stolz Criterion. *Let (x_n) and (y_n) be real sequences. Suupose that (y_n) is monotonic and $y_n \neq 0$ for all n. If $\lim_n (x_{n+1}-x_n)/(y_{n+1}-y_n) = L \in \mathbf{R} \cup \{\pm\infty\}$ exists, then $\lim_n x_n/y_n = L$ provided either $\lim_n x_n = \lim_n y_n = 0$ or $\lim_n y_n = \pm\infty$.*

Lemma 14. *If $\mu \in M(0, 1)$ and $x \in \mathbf{R}\backslash\operatorname{supp}\mu$ then*

$$\lim_n \frac{p_n^2(x)}{K_{n-1}(x, x)} = \lim_n \frac{p_n(x)\, p_n'(x)}{K_{n-1}^{(0,1)}(x, x)} = \lim_n \frac{p_n'^2(x)}{K_{n-1}^{(1,1)}(x, x)} = \varphi^2(x) - 1 \tag{4.6}$$

and

$$\lim_n n^{-\alpha} K_n(x, x) = +\infty \quad \textit{for every} \quad \alpha > 0 \tag{4.7}$$

Proof. If $x \in \mathbf{R}\backslash \operatorname{supp} \mu$, $(K_n(x, x))$ and $(K_n^{(1,1)}(x, x))$ are increasing sequences which tend to infinity. The same is true for $(K_n^{(0,1)}(x, x))$ whenever $x \in (0, +\infty)\backslash \operatorname{supp} \mu$. If $x \in (-\infty, 0)\backslash \operatorname{supp} \mu$, it is clear that $(K_n^{(0,1)}(x, x))$ is a decreasing sequence which tends to $-\infty$. By the Stolz criterion and (4.1), (4.6) follows.

On the other hand, it suffices to prove (4.7) for α a positive integer. Then, by Stolz criterion and (4.6),

$$\lim_n n^{-\alpha} K_n(x, x) = \lim_n \frac{p_n^2(x)}{n^\alpha - (n-1)^\alpha} = [\varphi^2(x) - 1] \lim_n \frac{K_{n-1}(x, x)}{n^{\alpha - 1}}$$

By applying this procedure $(\alpha - 1)$ times, (4.7) follows.

Remark. The asymptotic formula $\lim_n p_n^2(x)/K_{n-1}(x, x) = \varphi^2(x) - 1$ was proved by Nevai [17, Theorem 11, p. 31] using different techniques.

We are now ready to deduce our main results. A first one is an asymptotic expression for the ratio $(Q_n^\lambda, Q_n^\lambda)/\|P_n\|_\mu^2$, where (Q_n) is the SMOP associate with the inner product (2.7). In what follows, we write $\|Q_2^\lambda\|^2 = (Q_n^\lambda, Q_n^\lambda)$.

PROPOSITION 15. *Let $\mu \in M(0, 1)$ and $c \in \mathbf{R}\backslash \operatorname{supp} \mu$, then*

$$\lim_n \frac{\|Q_n^\lambda\|}{\|P_n\|_\mu} = |\varphi(c)|^{\text{rank A}} \tag{4.8}$$

Proof. From (2.2) and (2.8) we get $\|Q_n^\lambda\|^2/\|P_n\|_\mu^2 = \det(I + A\mathbb{H}_n)/\det(I + A\mathbb{H}_{n-1}) = \Delta_n(\lambda)/\Delta_{n-1}(\lambda)$.

It is clear, from (2.12), that $(\Delta_n(\lambda))$ is an increasing sequence which tends to infinity; so, by Stolz criterion, $\lim_n \|Q_n^\lambda\|^2/\|P_n\|_\mu^2 = \lim_n (\Delta_n(\lambda) - \Delta_{n-1}(\lambda))/(\Delta_{n-1}(\lambda) - \Delta_{n-2}(\lambda))$ where, by (2.12) and Lemma 2.1 in [2], straightforward calculations lead to

$$\begin{aligned} \Delta_n(\lambda) - \Delta_{n-1}(\lambda) = n^2 p_{n-1}'^2(c) \Bigg(& \frac{2\lambda}{n} \frac{p_{n-1}'(c)}{n p_{n-1}(c)} + \frac{M}{n^2} + N \frac{p_{n-1}'^2(c)}{n^2 p_{n-1}^2(c)} \\ & + (MN - \lambda^2) \frac{p_{n-2}^2(c; \mu_2)}{n^2} \left[1 + \frac{K_{n-2}(c, c)}{p_{n-1}^2(c)} + \frac{K_{n-3}(c, c; \mu_2)}{p_{n-2}^2(c; \mu_2)} \right] \Bigg) \end{aligned} \tag{4.9}$$

Whenever rank $A = 2$, on the right hand side of (4.9) the main contribution to the limit is the last term, so by (4.1) and (4.6), $\lim_n \|Q_n^\lambda\|^2/\|P_n\|_\mu^2 = \lim_n (p_{n-1}^2(c)/p_{n-2}^2(c))\ (p_{n-2}^2(c; \mu_2)/p_{n-3}^2(c; \mu_2)) = \varphi^4(c)$.

If rank $\text{A} = 1$, by analyzing separately the cases $N \neq 0$ and $N = 0$ (which implies $\lambda = 0$ and $M \neq 0$), from (4.1) and (4.2), we get $\lim_n \|Q_n^\lambda\|^2/\|P_n\|_\mu^2 = \lim_n p_{n-1}^2(c)/p_{n-2}^2(c) = \varphi^2(c)$ and the proof is complete. ∎

Denote by (q_n^λ) the sequence of orthonormal polynomials for the inner product (2.7). Since an asymptotic expression for the ratio $q_n^\lambda(x)/p_n(x)$ will be deduced from formula (2.13), we need to know the asymptotic behaviour of the coefficients α_n^λ and β_n^λ.

PROPOSITION 16. *Suppose* $\mu \in M(0, 1)$ *and* $c \in \mathbf{R}\backslash \operatorname{supp} \mu$. *Let* α_n^λ *and* β_n^λ *be the coefficients in formula* (2.13). *Then*

(a) *If* rank $\mathrm{A} = 2$, $\lim_n \alpha_n^\lambda = \lim_n \beta_n^\lambda = 1$

(b) *If* rank $\mathrm{A} = 1$, $\lim_n \alpha_n^\lambda = 1$ *and* $\lim_n \beta_n^\lambda = 0$

Proof. By (2.15) and Lemma 5 we can write:

$$\beta_n^\lambda = \frac{\|P_n\|_\mu}{\|P_{n-1}(.;\mu_2)\|_{\mu_2}} \frac{np_n(c)\, K_{n-2}(c, c; \mu_2)}{p_{n-1}(c; \mu_2)\, \Delta_n(\lambda)} \left[N \frac{p_n'(c)}{np_n(c)} + \frac{\lambda}{n} \right]$$
$$+ (MN - \lambda^2) \frac{K_{n-1}(c, c)\, K_{n-2}(c, c; \mu_2)}{\Delta_n(\lambda)} \tag{4.10}$$

Taking into account that $\det \mathbb{H}_{n-1} = K_{n-2}(c,c; \mu_2)\, K_{n-1}(c, c)$, from (2.12), (4.2) and Lemma 14 we find that, when rank $\mathrm{A} = 2$,

$$\lim_n \frac{1}{\Delta_n(\lambda)} [(MN - \lambda^2)\, K_{n-2}(c, c; \mu_2)\, K_{n-1}(c, c)] = 1 \tag{4.11}$$

and so, by letting n tend to infinity in (4.10) and using (4.2) and (4.11), $\lim_n \beta_n^\lambda = 1$ follows.

On the other hand, Lemma 5 allows us to write (2.14) in the form

$$\alpha_n^\lambda = \Delta_n(\lambda)^{-1} \left(MK_{n-1}(c, c) + N \frac{p_n'(c)}{p_n(c)} K_{n-1}^{(0,1)}(c, c) \right.$$
$$\left. + \lambda \left[\frac{p_n'(c)}{p_n(c)} K_{n-1}(c, c) + K_{n-1}^{(0,1)}(c, c) \right] + (MN - \lambda^2) \det \mathbb{H}_{n-1} \right) \tag{4.12}$$

(4.11), (4.6) and (4.7) imply that, for rank $\mathrm{A} = 2$, the only contribution in the limit is the last term in (4.12) and hence $\lim_n \alpha_n^\lambda = 1$

Next, suppose rank $\mathrm{A} = 1$. Then

$$\Delta_n(\lambda) = 1 + MK_{n-1}(c, c) + NK_{n-1}^{(1,1)}(c, c) + 2\lambda K_{n-1}^{(0,1)}(c, c)$$

and in (4.10) the last term vanishes. From (4.2) and (4.6), whether $N \neq 0$ or $N = 0$, we find that $\lim_n \beta_n^\lambda = 0$. Formula (4.12) together with the assumption rank $\mathrm{A} = 1$ and formula (4.6) yield $\lim_n \alpha_n^\lambda = 1$. ∎

We are ready to prove:

PROPOSITION 17. *Let* $\mu \in M(0,1)$ *and* $c \in \mathbf{R} \backslash \operatorname{supp} \mu$. *Then for all* $j = 0, 1, \ldots$

$$\lim_n \frac{(q_n^\lambda)^{(j)}(x)}{p_n^{(j)}(x)} = \left[\frac{(\varphi(x) - \varphi(c))^2}{2(x-c)\,\varphi(x)\,|\varphi(c)|}\right]^{\operatorname{rank} A} \tag{4.13}$$

uniformly on compact subsets of $(\mathbf{C} \backslash \operatorname{supp} \mu) \backslash \{c\}$ *and*

$$\lim_n \frac{(q_n^\lambda)^{(j)}(c)}{p_n^{(j)}(c)} = 0 \tag{4.14}$$

Proof. We write (2.13), in the form

$$\frac{q_n^\lambda(x)}{p_n(x)} = \frac{\|P_n\|_\mu}{\|Q_n^\lambda\|} \left[(1-\alpha_n^\lambda) + (\alpha_n^\lambda - \beta_n^\lambda)\frac{(x-c)\,P_{n-1}(x;\mu_2)}{P_n(x)} + \beta_n^\lambda \frac{(x-c)^2\,P_{n-2}(x;\mu_4)}{P_n(x)}\right] \tag{4.15}$$

From (4.1), (4.3) and (4.5) we obtain

$$\lim_n \frac{(x-c)\,P_{n-1}(x;\mu_2)}{P_n(x)} = \frac{(\varphi(x)-\varphi(c))^2}{2(x-c)\,\varphi(x)} \tag{4.16}$$

uniformly on compact subsets of $(\mathbf{C} \backslash \operatorname{supp} \mu) \backslash \{c\}$.

As an easy consequence of the above we get

$$\lim_n \frac{(x-c)^2\,P_{n-2}(x;\mu_4)}{P_n(x)} = \left(\frac{(\varphi(x)-\varphi(c))^2}{2(x-c)\,\varphi(x)}\right)^2 \tag{4.17}$$

uniformly on compact subsets of $(\mathbf{C} \backslash \operatorname{supp} \mu) \backslash \{c\}$. The asymptotic result (4.13) for $j=0$ follows by letting n tend to infinity in (4.15), using Propositions 15 and 16 and either (4.16) or (4.17).

Now we can differentiate both sides of

$$\lim_n \frac{q_n^\lambda(x)}{p_n(x)} = \left[\frac{(\varphi(x) - \varphi(c))^2}{2(x-c)\,\varphi(x)\,|\varphi(c)|}\right]^{\operatorname{rank} A}$$

(see [19, Theorem 10.28]). Hence

$$\lim_n \left(\frac{q_n^\lambda(x)}{p_n(x)}\right)' = \lim_n \frac{p_n'(x)}{p_n(x)} \left[\frac{q_n^{\lambda\prime}(x)}{p_n'(x)} - \frac{q_n^\lambda(x)}{p_n(x)}\right] = f'(x)$$

uniformly on compact subsets of $(\mathbf{C}\backslash\operatorname{supp}\mu)\backslash\{c\}$, where the definition of f is obvious. From (4.2) and (4.13) for $j=0$, it follows that (4.13) is true for $j=1$.

The proof of (4.13) is completed by induction using the scheme employed in passing to the first derivative and taking into account (4.2).

Formula (4.14) follows directly from (4.13). ∎

Remark. For $j=0$, this proposition includes the results for the cases $M\neq 0$, $N=0=\lambda$ (see [17, Lemma 16 on p. 132]) and $N\neq 0$, $M=0=\lambda$ (see [15, Theorem 2]).

Also, this asymptotic result has been obtained in [11] by using different techniques.

Proposition 17 tells us that the point c, where the derivatives in Sobolev-type inner product are evaluated, is the zero of the limit function in the relative asymptotic behaviour. Next we are going to obtain the masses in the discrete part of the inner product.

PROPOSITION 18. *Let $\mu\in M(0,1)$ and $c\in\mathbf{R}\backslash\operatorname{supp}\mu$. The following assertions are valid:*

(a) *If* rank $\mathrm{A}=2$ *and* $\lambda\neq 0$, *then* $\lim_n p_n'(c)\, q_n^{\lambda'}(c)/n^2 = 4\lambda\sqrt{c^2-1}/(MN-\lambda^2)$

(b) *If* rank $\mathrm{A}=2$ *and* $\lambda=0$, *then* $\lim_n p_n'(c)\, q_n^{\lambda'}(c)/n = 4N^{-1}(c^2-1)$

(c) *If* rank $\mathrm{A}=1$ *and* $MN\lambda\neq 0$, *then* $\lim_n p_n'(c)\, q_n^{\lambda'}(c)/p_n^2(c) = \lambda\operatorname{sgn} c/2N(c^2-1)$

(d) *If* rank $\mathrm{A}=1$ *and* $M=\lambda=0$, *then* $\lim_n p_n'(c)\, q_n^{\lambda'}(c) = 2N^{-1}|\sqrt{c^2-1}|$

(e) *If* rank $\mathrm{A}=1$ *and* $N=\lambda=0$, *then* $\lim_n p_n(c)\, q_n(c) = 2M^{-1}|\sqrt{c^2-1}|$.

Proof. Formula (2.11) for orthonormal polynomials gives

$$\frac{\|Q_n^\lambda\|}{\|P_n\|_\mu}\, q_n^{\lambda'}(c) = \frac{p_n'(c)}{\Delta_n(\lambda)} + \frac{\|P_{n-1}(\cdot;\mu_2)\|_{\mu_2}}{\|P_n\|_\mu}\frac{K_{n-1}(c,c)}{\Delta_n(\lambda)}\,(Mp_{n-1}(c;\mu_2)+\lambda p_{n-1}'(c;\mu_2))$$

(a) Suppose rank $\mathrm{A}=2$ and $\lambda\neq 0$. From (4.2), (4.6) and (4.11) we find that

$$\lim_n \frac{q_n^{\lambda'}(c)\, p_n'(c)}{n^2} = \lim_n \frac{\|P_n\|_\mu}{\|Q_n^\lambda\|}\frac{\|P_{n-1}(\cdot;\mu_2)\|_{\mu_2}}{\|P_n\|_\mu}\frac{\lambda p_{n-1}'(c;\mu_2)\, p_n'(c)}{(MN-\lambda^2)\, n^2 K_{n-2}(c,c;\mu_2)}$$

Using (4.1)–(4.6) and Proposition 15 we obtain

$$\lim_n \frac{q_n^{\lambda'}(c)\, p_n'(c)}{n^2} = \frac{2\lambda}{MN-\lambda^2}\, \frac{\varphi^2(c)-1}{\varphi(c)} = \frac{4\lambda}{MN-\lambda^2}\sqrt{c^2-1}$$

where the last equality follows from the function φ.

(b) If rank A $=2$ and $\lambda=0$, in a similar way we have

$$\lim_n \frac{q_n^{\lambda'}(c)\, p_n'(c)}{n} = \lim_n \frac{\|P_n\|_\mu}{\|Q_n^\lambda\|}\, \frac{\|P_{n-1}(.;\mu_2)\|_{\mu_2}}{\|P_n\|_\mu}\, \frac{p_{n-1}(c;\mu_2)\, p_n'(c)}{NnK_{n-2}(c,c;\mu_2)}$$

and by the same reasoning as before, it follows that $\lim_n q_n^{\lambda'}(c)\, p_n'(c)/n = 4N^{-1}(c^2-1)$.

(c) It suffices to proceed as in the two previous cases, taking into account that if rank A $=1$ with $MN\lambda \neq 0$, then

$$\Delta_n(\lambda) = 1 + MK_{n-1}(c,c) + 2\lambda K_{n-1}^{(0,1)}(c,c) + NK_{n-1}^{(1,1)}(c,c)$$

and hence $\lim_n N\,\Delta_n(\lambda)^{-1}\, K_{n-1}^{(1,1)}(c,c) = 1$. Since

$$\lim_n \frac{q_n^{\lambda'}(c)\, p_n'(c)}{p_n^2(c)} = \frac{\lambda}{N} \lim_n \frac{\|P_n\|_\mu}{\|Q_n^\lambda\|}\, \frac{\|P_{n-1}(.;\mu_2)\|_{\mu_2}}{\|P_n\|_\mu}\, \frac{p_{n-1}'(c;\mu_2)\, p_n'(c)\, K_{n-1}(c,c)}{p_n^2(c)\, K_{n-1}^{(1,1)}(c,c)}$$

the result follows from (4.1)–(4.6) and Proposition 15.

(d) In a similar way we can find this asymptotic formula, which was already given in [15, formula (22) on p. 199].

(e) Whenever rank A $=1$ and $N=\lambda=0$, formulas (2.8) and (2.10) give

$$q_n^\lambda(c)\, p_n(c) = \frac{\|P_n\|_\mu}{\|Q_n^\lambda\|}\, \frac{p_n^2(c)}{1+MK_{n-1}(c,c)}$$

and the result follows immediately. This result can also be found in [17, Lemma 16]. ∎

The asymptotic behaviour of some expressions involving either $q_n^{\lambda'}(c)\, p_n'(c)$ or $q_n^\lambda(c)\, p_n(c)$ gives us information about the matrix A in the inner product.

Corollary 19. *Let q_n^λ and p_n the orthonormal polynomials associated with the product (2.7) and the measure μ, respectively. Suppose $\mu \in M(0,1)$ and $c \in \mathbf{R}\backslash \operatorname{supp} \mu$. Then*

(a) rank A $=2$ *and* $\lambda \neq 0$ *if and only if* $\lim_n q_n^{\lambda'}(c)\, p_n'(c)/n^2 \in \mathbf{R}\backslash\{0\}$

(b) rank A $=2$ *and* $\lambda=0$ *and only if* $\lim_n q_n^{\lambda'}(c)\, p_n'(c)/n \in \mathbf{R}\backslash\{0\}$

(c) rank A $=1$ *and* $MN\lambda \neq 0$ *if and only if* $\lim_n q_n^{\lambda'}(c)\, p_n'(c)/p_n^2(c) \in \mathbf{R}\backslash\{0\}$

(d) rank A $=1$ *and* $M=\lambda=0$ *if and only if* $\lim_n q_n^{\lambda'}(c)\, p_n'(c) \in \mathbf{R}\backslash\{0\}$

(e) rank A $=1$ *and* $N=\lambda=0$ *if and only if* $\lim_n q_n^{\lambda}(c)\, p_n(c) \in \mathbf{R}\backslash\{0\}$

Proof. Because of Proposition 18, in all these five cases, it suffices to prove the corresponding sufficient condition.

Firstly, let us note that when rank A $=1$ and $N=\lambda=0$ it can be deduced that $\lim_n p_n^{-2}(c)\, q_n^{\lambda'}(c)\, p_n'(c)=\infty$ and hence $\lim_n n^{-\alpha} q_n^{\lambda'}(c)\, p_n'(c)=\infty$ for $\alpha=0, 1, 2$.

(a) Assume that $\lim_n n^{-2} q_n^{\lambda'}(c)\, p_n'(c) \in \mathbf{R}\backslash\{0\}$. Then the case rank A $=1$ and $N=\lambda=0$ is excluded. On the other hand, it is obvious that in b), c) and d) the corresponding limit is either infinity or zero. Thus the only one possibility is rank A $=2$ and $\lambda \neq 0$.

The remaining cases (b), (c), and (d) can be proved in the same way.

(e) From (2.13), we have $q_n^{\lambda}(c)\, p_n(c)=(\|P_n\|_\mu (1-\alpha_n^{\lambda})\, p_n^2(c))/\|Q_n^{\lambda}\|$. Now, it is not difficult to show that $N \neq 0$ implies $\lim_n q_n^{\lambda}(c)\, p_n(c)=\infty$. So, $\lim_n q_n^{\lambda}(c)\, p_n(c) \in \mathbf{R}\backslash\{0\}$ gives $N=0$ and therefore rank A $=1$ and $\lambda=0$. ∎

Next we analyze the asymptotic behaviour of the zeros of Q_n^{λ} off supp μ. Recall that $(\xi_{nk})_{k=1}^n$ are the zeros of Q_n^{λ} ordered by increasing size.

Let $\mu \in M(0, 1)$ and suppose $c \geqslant 1$, then Proposition 9 assures that for all sufficiently large n, Q_n^{λ} has real and simple zeros, $\xi_{nn}>c$ and the other zeros belong to $(-1, c)$.

PROPOSITION 20. *If* $N=0$, *then for all* $\alpha \in (0, 1)$, $\xi_{nn}-c=o(n^{-\alpha})$ *when* n *tends to infinity*.

Proof. If $N \neq 0$, from Lemma 5(ii), $\lim_n Q_n^{\lambda'}(c)/n^\alpha Q_n^{\lambda}(c)=0$ holds for all $\alpha \geqslant 0$, where formulas (4.2) and (4.6) have been used. Otherwise, as

$$\frac{Q_n^{\lambda'}(c)}{Q_n^{\lambda}(c)}=\sum_{k=1}^{n} \frac{1}{c-\xi_{nk}}$$

and $(\xi_{nk})_{k=1}^n \subset (-1, c)$, we have $1/(\xi_{nn}-c)>(n-1)/(c+1)-[Q_n^{\lambda'}(c)/Q_n^{\lambda}(c)]$.
Therefore, $\xi_{nn}-c=o(n^{-\alpha})$ for all $\alpha \in (0, 1)$. ∎

Let us note that, when $N \neq 0$, c attracts the greatest zero of Q_n^{λ}.

On the other hand, we know that each polynomial Q_n^{λ} has at most two zeros off $(-1, 1)$. We are going to show that when rank A $=2$ and n is large enough Q_n^λ has precisely two zeros off $(-1, 1)$ and both are attracted by c.

PROPOSITION 21. *Let* rank $A = 2$. *Then there exists a positive integer* n_0 *such that for every* $n \geqslant n_0$, Q_n^λ *has two zeros off* $(-1, 1)$. *Besides,* $1 < \xi_{n,n-1} < c < \xi_{n,n}$ *and* $\lim_n \xi_{n,n-1} = \lim_n \xi_{n,n} = c$.

Proof. From formulas (4.10) and (4.12) it can be deduced that

$$\lim_n (1-\alpha_n^\lambda)\, P_n(c) = 0 = \lim_n (\alpha_n^\lambda - \beta_n^\lambda)\, P_{n-1}(c; \mu_2)$$

which implies that, for $x \in [1, c)$,

$$\lim_n (1-\alpha_n^\lambda)\, P_n(x) = 0 = \lim_n (\alpha_n^\lambda - \beta_n^\lambda)\, P_{n-1}(x; \mu_2)$$

Recall that when rank $A = 2$, then $\lim_n \beta_n^\lambda = 1$. Thus if we evaluate (2.13) at $x = 1$ and if we use the above results then we have $\lim_n Q_n^\lambda(1) > 0$. As $Q_n^\lambda(c) < 0$ for all n large enough (see Proposition 9), we find that $1 < \xi_{n,n-1} < c < \xi_{n,n}$ for all sufficiently large n.

Let $\varepsilon \in \mathbf{R}$ with $0 < \varepsilon < c - 1$. By repeating the above argument we obtain that $\lim_n Q_n^\lambda(c-\varepsilon) > 0$ and so there exists a positive integer n_1 such that $c - \varepsilon < \xi_{n,n-1} < c$ for all $n \geqslant n_1$ and hence $\lim_n \xi_{n,n-1} = c$.

Remark. Let us note that when rank $A = 1$ and $\lambda \geqslant 0$, because of Proposition 13, Q_n^λ has only one zero off $(-1, 1)$. The question whether this result is true for rank $A = 1$ and $\lambda < 0$ remains open.

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